ON NONLINEAR REFLECTION OF WEAK SHOCK WAVES

(O NELINEINOM OTRAZHENII SLABYKH UDARNYKH VOLN)

PMM Vol.22, No.5, 1958, pp.586-599

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(Received 20 May 1958)

In this paper certain physical conditions are considered which permit significant simplifications to be made in the equations of gas dynamics describing the non-steady flows with small but sharp changes of the parameters of the medium. The mathematical approximations are based on the fact that the pressure variations in the stream take place in a small region adjacent to the shock wave front. Such flows are named "short waves". Exact particular solutions of the derived nonlinear differential equations are obtained. These solutions are then used for the approximate solution of a problem of nonlinear reflection of a shock wave from a perfectly rough wall. The boundary conditions at the front of one or several shock waves can be satisfied with sufficient accuracy by proper choice of the constants contained in the particular solutions. The boundary conditions at the wall are satisfied automatically.

In studying the propagation of the waves with small relative excess pressure p/P_0 (where P_0 is the initial pressure in the undisturbed medium and p is the variation of pressure) usually the acoustical equations are made use of. In that case the propagation velocity of disturbances is considered to be constant and equal to the velocity of sound in the undisturbed medium.

It is observed, however, that some phenomena in spite of the small relative excess pressures in the waves are determined entirely by the dependence of the velocity of propagation of disturbances on the magnitude of excess pressure. Thus the laws of extinction of shock waves at great distances from the place of explosion are determined basically by this dependence [1,2]. This dependence introduces considerable changes into the picture of shock wave reflection from a free surface created by the explosion of a charge near the surface [3]. To these phenomena belongs also the reflection of the shock wave from a rough surface for small angles between the shock wave front and the normal to the wall.

Investigation of these phenomena cannot be based on the linear acoustic equations. The fact is that in spite of the smallness of the relative excess pressure the variation of the pressure in the cases indicated takes place in a small region. Therefore, the pressure gradients are large and the variations of the velocity of disturbance propagation, due to pressure, are significant. Such waves with large pressure gradients near the shock front may be called "short waves".

1. Short waves. 1. In the case of weak shock waves the process of compression can be regarded as adiabatic with great accuracy and the law of compression for water may be taken in the form

$$p = P_0\,[(\rho/\rho_0)^n - 1] \tag{1.1}$$

where ρ_0 is the initial density of the water, ρ the actual density, p the pressure, and P_0 and n are constants. For temperatures of the order of $+15°C$ $P_0 \approx 3000\ \mathrm{kg/cm^2}$ and $n \approx 7$.

Relation (1.1) is also valid for air, if P_0 stands for initial air pressure, p excess pressure and $n = 1.4$.

2. The equations of dynamics in a spherical coordinate system with symmetry around the axis $\theta = 0$ may be written in the form

$$\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \vartheta} - \frac{v^2}{r} + \frac{2}{n-1}\,a\frac{\partial a}{\partial r} = 0 \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \vartheta} + \frac{uv}{r} + \frac{2}{n-1}\,\frac{a}{r}\frac{\partial a}{\partial \vartheta} = 0 \end{aligned} \tag{1.2}$$

where t is the time, r is the distance, u and v are the projections of the velocity vector q on the direction of the radiusvector and the perpendicular to it, a is the velocity of sound. For the velocity of sound we have

$$a = a_0\,(\rho/\rho_0)^{1/2\,(n-1)}, \qquad a_0 = \left(\frac{nP_0}{\rho_0}\right)^{1/2} \tag{1.3}$$

where a_0 is the velocity of sound in an undisturbed medium. The equation of continuity using these variables will have the form:

$$\frac{\partial a}{\partial t} + u\frac{\partial a}{\partial r} + \frac{v}{r}\frac{\partial a}{\partial \vartheta} + \frac{n-1}{2}\,\frac{a}{r}\left(r\frac{\partial u}{\partial r} + \frac{\partial v}{\partial \vartheta} + 2u + v\cot\vartheta\right) = 0 \tag{1.4}$$

3. Assume

$$u = a_0 M, \qquad v = a_0 V \tag{1.5}$$

For short shock waves M and V are small quantities. For the investigation of short waves it is convenient to introduce the following variables:

$$r = a_0 t(1+\Delta), \qquad a = a_0\left(1+\frac{n-1}{2}M_0\alpha\right), \qquad \tau = \ln t$$
$$M = M_0\mu, \qquad V = V_0\sqrt{\frac{n+1}{2}}\,\nu, \qquad \vartheta = \theta_0\sqrt{\frac{n+1}{2}}\,Y, \qquad \Delta = \frac{n+1}{2}M_0\delta \tag{1.6}$$

where M_0, V_0 and θ_0 are characteristic values of M, V and θ, and M_0 and V_0 are small compared to unity, but μ, ν, a and δ are of the order of unity. In such a manner the wavelength in the radial direction is assumed to be a small quantity, of the order of M_0.

Using the variables introduced and disregarding quantities small compared to unity in the coefficients, one may write equations (1.3), (1.4) in the form:

$$\frac{\partial\alpha}{\partial\delta} - \frac{\partial\mu}{\partial\delta} + \frac{n+1}{2}\left(M_0\frac{\partial\mu}{\partial\tau} + \frac{M_0V_0}{\theta_0}\nu\frac{\partial\mu}{\partial Y} - \frac{n+1}{2}V_0^2\nu^2\right) = 0$$
$$\frac{M_0^2}{V_0\theta_0}\frac{\partial\alpha}{\partial Y} - \frac{\partial\nu}{\partial\delta} + \frac{n+1}{2}\left(M_0\frac{\partial\nu}{\partial\tau} + \frac{V_0M_0}{\theta_0}\nu\frac{\partial\nu}{\partial Y} + M_0^2\mu\nu\right) = 0$$
$$\frac{\partial\mu}{\partial\tau} + \frac{\partial\alpha}{\partial\tau} + \frac{2}{n+1}\left(\mu + \frac{n+1}{2}\alpha - \frac{n+1}{2}\delta\right)\left(\frac{\partial\mu}{\partial\delta} + \frac{\partial\alpha}{\partial\delta}\right) + \frac{V_0}{M_0\theta_0}\frac{\partial\nu}{\partial Y} + \tag{1.7}$$
$$+ 2\mu + \frac{V_0}{\theta_0}\nu\left(\frac{\partial\mu}{\partial Y} + \frac{\partial\alpha}{\partial Y}\right) - \frac{n+1}{2}\frac{V_0^2}{M_0}\nu^2 + \sqrt{\frac{n+1}{2}}\frac{V_0}{M_0}\nu\cot\left(\sqrt{\frac{n+1}{2}}\theta_0 Y\right) = 0$$

In the last equation of (1.7) the difference $\partial\mu/\partial\delta - \partial\alpha/\partial\delta$ was replaced by its equivalent expression from the first equation of this system. From this equation it follows that the derivatives $\partial\mu/\partial\tau$ and $\partial\alpha/\partial\tau$ are of an order not greater than unity. Besides,

$$\frac{V_0}{M_0\theta_0} \ll 1 \tag{1.8}$$

From the second equation of (1.7) it follows then that

$$\frac{M_0^2}{V_0\theta_0} \sim 1 \tag{1.9}$$

if $\theta_0 \sim 1$, then from the relation (1.9) it is seen that $V_0 \sim M_0^2$. If θ_0 is a small quantity, then from formulas (1.8) and (1.9) we will obtain

$$\theta_0 \sim \sqrt{M_0}, \qquad V_0 \sim M_0\sqrt{M_0} \tag{1.10}$$

In all the cases the first of the equations (1.7), after neglecting all quantities of small magnitude, will take the form:

$$\frac{\partial\alpha}{\partial\delta} = \frac{\partial\mu}{\partial\delta}$$

Integrating this equation and taking into consideration that only those shock waves will be considered in the future, for which the excess pressure and the velocity of the particles will be equal to zero just in front of the wave front, we will obtain $\alpha = \mu$.

From equations (1.1) and (1.3) follows that $\alpha = p/nP_0M_0$ and, consequently

$$M = p/nP_0 \qquad (1.11)$$

In considering the propagation of weak short waves in water, if we limit ourselves to pressures not exceeding 200-300 atm, then there correspond to these pressures numbers M less than 0.01-0.05. For weak short waves in air it is necessary to limit the excess pressures to 0.15-0.20 atm for $P_0 = 1$ atm. Then the maxima of the numbers M will be 0.10-0.15.

Let us assume now $M_0^2 = V_0\theta_0$. Because the quantity V_0M_0/θ_0 is always small, the second equation of (1.7) after neglecting quantities of small magnitude will have the form:

$$\frac{\partial \nu}{\partial \delta} - \frac{\partial \mu}{\partial Y} = 0 \qquad (1.12)$$

The above relationship expresses the condition that the flow is irrotational.

If $\theta_0 \sim 1$, then $V_0 \sim M_0^2$ and the last equation of (1.7) will take the form:

$$\frac{\partial \mu}{\partial \tau} + (\mu - \delta)\frac{\partial \mu}{\partial \delta} + \mu = 0 \qquad (1.13)$$

The system of equations (1.11) and (1.13) is identical to the system of equations for short waves in the case of central symmetry [2]. In this manner, when the angle θ is varies in a finite interval, i.e. when the derivatives in the direction normal to the radial direction are small, the motion of the wave along any ray is independent of the motion along the neighboring rays. The general integral of the system (1.12) and (1.13) is easily determined.

4. Let θ_0 be a small quantity, i.e. let the wave be "short" not only in the direction of the radius but also in the direction perpendicular to it. Then in relation to (1.10) we assume $\theta_0 = \sqrt{M_0}$ and $V_0 = M_0\sqrt{M_0}$.

If the variations in the parameters of a stream take place in the vicinity of the axis $\theta = 0$, then the third equation of (1.7) will take the form:

$$\frac{\partial \mu}{\partial \tau} + (\mu - \delta)\frac{\partial \mu}{\partial \delta} + \frac{1}{2}\frac{\partial \nu}{\partial Y} + \mu + \frac{\nu}{2Y} = 0 \qquad (1.14)$$

If the region of sharp variations of the flow parameters is situated near the finite angle $\theta = \theta_*$, then the last equation is simplified:

$$\frac{\partial \mu}{\partial \tau} + (\mu - \delta)\frac{\partial \mu}{\partial \delta} + \frac{1}{2}\frac{\partial \nu}{\partial Y} + \mu = 0 \qquad (1.15)$$

In a similar manner the system of equations of short waves may be obtained in the case of plane flows also. For the plane waves in cylindrical coordinate system, we have

$$\frac{\partial\mu}{\partial\tau}+(\mu-\delta)\frac{\partial\mu}{\partial\delta}+\frac{1}{2}\frac{\nu}{\partial Y}+\frac{1}{2}\mu=0$$
$$\frac{\partial\nu}{\partial\delta}-\frac{\partial\mu}{\partial Y}=0, \qquad M=\frac{p}{nP_0} \tag{1.16}$$

If the wave length is of an order of magnitude less than M_0, and the derivatives of μ and ν with respect to δ and Y are large, then the equations (1.15) and (1.16) are simplified still more; in the case of "very short waves" we have

$$(\mu-\delta_0)\frac{\partial\mu}{\partial\delta}+\frac{1}{2}\frac{\partial\nu}{\partial Y}=0, \qquad \frac{\partial\nu}{\partial\delta}-\frac{\partial\mu}{\partial Y}=0 \tag{1.17}$$

where δ_0 is constant. The system of equations (1.17) is analogous to the equations which describe sonic gas flows [4].

If the flow is self-similar, i.e. if it does not depend on τ, then for spherical waves we have

$$(\mu-\delta)\frac{\partial\mu}{\partial\delta}+\frac{1}{2}\frac{\partial\nu}{\partial Y}+\mu=0, \qquad \frac{\partial\nu}{\partial\delta}-\frac{\partial\mu}{\partial Y}=0 \tag{1.18}$$

Correspondingly, for plane waves we obtain

$$(\mu-\delta)\frac{\partial\mu}{\partial\delta}+\frac{1}{2}\frac{\partial\nu}{\partial Y}+\frac{1}{2}\mu=0, \qquad \frac{\partial\nu}{\partial\delta}-\frac{\partial\mu}{\partial Y}=0 \tag{1.19}$$

The systems of equations (1.18) and (1.19) will be of the hyperbolic type for $\delta > \mu$ and of the elliptical type in the reverse case. The equation of the "sonic line" will be, correspondingly, $\delta = \mu$.

2. Some particular solutions of self-similar equations. 1. Let us construct particular solutions of the systems of equations (1.18) and (1.19) for which $\nu = 0$ for $Y = 0$. Let us transform these equations beforehand, assuming μ and Y to be independent variables, and δ and ν to be the desired functions. The system of equations (1.18) will then take the form:

$$\frac{\partial\nu}{\partial\mu}+\frac{\partial\delta}{\partial Y}=0, \qquad \mu\frac{\partial\delta}{\partial\mu}+\frac{1}{2}\left[\frac{\partial\nu}{\partial Y}\frac{\partial\delta}{\partial\mu}+\left(\frac{\partial\nu}{\partial\mu}\right)^2\right]+\mu-\delta=0 \tag{2.1}$$

The transformed equations (1.19) are analogous to the equations (2.1). Let us look for particular solutions of the system (2.1) of the form

$$\nu=\varphi(\mu)Y, \qquad \delta=-\tfrac{1}{2}\varphi'(\mu)Y^2+F(\mu) \tag{2.2}$$

To determine the functions ϕ and F we have the system of equations

$$\varphi' - \tfrac{1}{2}\varphi\varphi'' + \varphi'^2 - \varphi''\mu = 0, \qquad \left(\mu + \tfrac{1}{2}\varphi\right)F' - F + \mu = 0 \tag{2.3}$$

For plane waves the analogous system has the form:

$$\varphi' - \tfrac{1}{2}\varphi\varphi'' + \varphi'^2 - \tfrac{1}{2}\mu\varphi'' = 0, \qquad \tfrac{1}{2}(\mu + \varphi)F' - F + \mu = 0 \tag{2.4}$$

The general solution of the system (2.3) will be

$$\varphi = \frac{A^2 - B^2 - \mu^2}{A + \mu}$$
$$F = \mu - \frac{1}{2B}[(\mu + A)^2 - B^2]\ln\frac{\mu + A - B}{\mu + A + B} + C[(\mu + A)^2 - B^2] \tag{2.5}$$

where A, B and C are arbitrary constants.

The solution of the system of equations (2.4) for the plane waves may be represented in the form

$$\varphi = \frac{1}{C'}\tan(C'\mu + A') - \mu, \quad F = B\sin^2(A' + C'\mu) + \frac{1}{2C'}\sin 2(C'\mu + A') + \mu \tag{2.6}$$

where A', B and C' are arbitrary constants. Assuming

$$C' = ic, \qquad A' = -iac + \tfrac{1}{2}\pi, \qquad B = -(1 + k)/c$$

let us present equations (2.6) in the form

$$\varphi = -\frac{1}{c}\coth c(a - \mu) - \mu, \quad F = \mu - \frac{1 + e^{-2c(a-\mu)}}{2c} - \frac{k}{2c}[1 + \cosh 2c(a - \mu)] \tag{2.7}$$

Assuming $C' = ic$, $A' = -iac$ and $B = (1 + k)/c$, we will write equations (2.6) in the form

$$\varphi = -\frac{1}{c}\tanh c(a - \mu) - \mu, \; F = \mu + \frac{1 - e^{2c(a-\mu)}}{2c} - \frac{k}{2c}[\cosh 2c(a - \mu) - 1] \tag{2.8}$$

2. Taking μ and ν to be independent variables, and the functions δ and Y to be unknown variables, from the system of equations (1.17) we obtain

$$(\mu - \delta_0)\frac{\partial Y}{\partial \nu} + \frac{1}{2}\frac{\partial \delta}{\partial \mu} = 0, \qquad \frac{\partial Y}{\partial \mu} - \frac{\partial \delta}{\partial \nu} = 0 \tag{2.9}$$

This system has a particular solution, analogous to the one obtained for equations (1.19) above:

$$\delta - \delta_0 = c^2[\tfrac{1}{2}\nu^2 + (\delta_0 - a)(a - \mu)^2 + \tfrac{2}{3}(a - \mu)^3], \; Y = -c^2(a - \mu)\nu \tag{2.10}$$

Another particular solution, which does not have a singularity on the line $\mu = \delta_0$, is given by the formula [4]

$$\mu - \delta_0 = \frac{1}{2\varepsilon}(\delta - \delta_0) - \frac{1}{4\varepsilon^2}Y^2, \qquad \nu = -\frac{1}{2\varepsilon^2}(\delta - \delta_0)Y + \frac{1}{12\varepsilon^3}Y^3 \quad (2.11)$$

where a, c, and ϵ are constants.

3. Boundary conditions at the front of a shock wave. 1. Let the shock wave have an excess pressure p at its front and be propagated in a medium with an initial excess pressure p_1 and with a particle velocity q_1 perpendicular to the front. Then the velocity of propagation of the shock wave front N and the velocity of the particles behind the front will be [3]

$$N = a_0\left(1 + \frac{n+1}{4}\frac{p}{P_0 n} + \frac{n-3}{4}\frac{p_1}{P_0 n}\right) + q_1, \qquad q = a_0\frac{p - p_1}{P_0 n} + q_1$$

If the normal to the shock wave front and, consequently, also the particle velocity vector make a small angle with the direction of the radius vector, then the projection of the particle velocity on both those directions may be considered to be equal to q. Therefore, if $q_1 = a_0 p_1 / P_0 n$, using notations of (1.5) we have

$$M = \frac{p}{P_0 n}, \qquad N = a_0\left[1 + \frac{n+1}{4}(M + M_1)\right] \quad (3.1)$$

2. Let the solution of the short wave equations be given by the functions

$$\delta = \delta(\mu, Y, \tau), \qquad \nu = \nu(\mu, Y, \tau) \quad (3.2)$$

Within the approximation considered here the projection of the particle velocity at the shock wave front and within the stream upon the radius vector (or number M) is related by the same equation (1.11) or (3.1). Therefore, as the solution (3.2) is known, it is easy to construct a differential equation which determines the location of the shock wave front, which bounds the zone of disturbed motion.

Let ψ be the angle between the normal to the shock wave front and the direction of the radius vector. We have

$$\psi = \frac{1}{r}\frac{\partial r}{d\theta} = \frac{d\Delta}{d\vartheta} = \sqrt{\frac{n+1}{2}}\frac{M_0}{\theta_0}\frac{d\delta}{dY} \quad (3.3)$$

For the projections of the velocity vector downstream of the front the equalities

$$u = q\cos\psi \approx q, \qquad v = q\sin\psi \approx u\psi \quad (3.4)$$

are valid. The velocity of propagation of the front in the direction of the radius vector is equal to

$$\frac{N}{\cos\psi} = a_0\left[1 + \frac{n+1}{4}(M + M_1) + \frac{\psi^2}{2}\right] \quad (3.5)$$

On the other hand, this velocity may be determined with the help of equations (3.2):

$$\frac{dr}{dt} = a_0\left(1 + \Delta + \frac{\partial\Delta}{\partial M}\frac{dM}{d\tau} + \frac{\partial\Delta}{\partial\tau}\right) \tag{3.6}$$

Thus, the equation of the shock wave front will be

$$\Delta + \frac{\partial\Delta}{\partial M}\frac{dM}{d\tau} + \frac{\partial\Delta}{\partial\tau} = \frac{n+1}{4}(M + M_1) + \frac{\psi^2}{2} \tag{3.7}$$

If the solution (3.2) does not depend on r, then from formula (3.7) follows

$$\Delta + \frac{\partial\Delta}{\partial M}\frac{dM}{d\tau} = \frac{n+1}{4}(M + M_1) + \frac{\psi^2}{2} \tag{3.8}$$

If the pressure at the wave front does not depend on time either, then the differential equation of the shock front is simplified still more and is written in the form:

$$\Delta = \tfrac{1}{4}(n+1)(M + M_1) + \tfrac{1}{2}\psi^2 \tag{3.9}$$

When θ_0 is a small quantity, we assume in accordance with (1.10)

$$\theta_0 = \sqrt{M_0}, \qquad V_0 = M_0\sqrt{M_0}$$

Equation (3.9), taking into consideration (3.3), may then be written in the form

$$\delta = \tfrac{1}{2}(\mu + \mu_1) + \tfrac{1}{2}(d\delta / dY)^2 \tag{3.10}$$

When changing to the Cartesian coordinate system, we have

$$x = a_0 t\,[1 + \tfrac{1}{2}(n+1)M_0 X] = r\cos\vartheta \approx a_0 t\,(1 + \Delta - \tfrac{1}{2}\vartheta^2) \tag{3.11}$$

$$y = a_0 t\sqrt{\frac{n+1}{2}}\,M_0 Y = r\sin\vartheta \approx a_0 t\vartheta$$

Consequently,

$$\delta = X + \tfrac{1}{2}Y^2 \tag{3.12}$$

The differential equation of the shock wave front in the Cartesian coordinate system is given by

$$dX / dY = -[Y + \sqrt{2\delta - (\mu + \mu_1)}] \tag{3.13}$$

3. Let us now take into consideration the condition of continuity of the projection of the velocity vector parallel to the front during transition through the shock wave. Let the medium before the wave front be at rest. Then the component of the velocity vector parallel to the front is equal to zero, and the values M and V, given by equations (3.2) at the front of the shock wave, must satisfy the relationship

$$M\psi + V = 0 \tag{3.14}$$

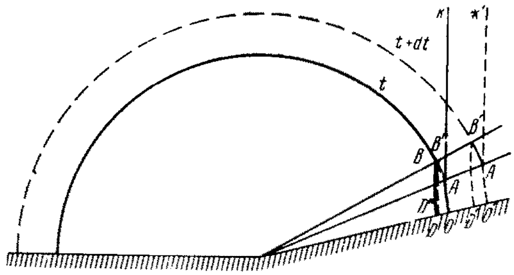

Fig. 1.

In pursuing our investigation we shall first try to satisfy condition (3.14) and an analogous condition at the front of the shock wave propagated through a medium already disturbed, and then check the degree to which that condition has been fulfilled. A certain integral relationship may be satisfied approximately instead of satisfying approximately the condition of continuity of the tangential component of the velocity vector.

Let us write in the form of an integral the law of conservation of the component of the momentum of the fluid in the direction perpendicular to the wall, for the region including the front of a shock wave.

Let us consider, for example, the flow that occurs in the case of the so-called irregular reflection of a shock wave from a rough wall (Fig. 1). The curve OA represents a Mach wave, AK the front of an oncoming wave, AB the front of a reflected wave and BD the line of equal pressure p. Let us consider the mass of the fluid occurring in the region $O'A'B'BD$ at time t. At this time let the front of the Mach wave occupy the position OA and the front of the reflected wave occupy the position AB. At time $t + dt$ these fronts will move to positions $O'A'$ and $A'B'$. The particles, lying on the line BD, will take the position $B''D''$.

Let u_1 denote the velocity of the particles behind the front of an oncoming wave, and p_1 the excess pressure. The component of the velocity vector perpendicular to the wall will be $u_1 a$, where a is the angle between the plane of the oncoming wave and the perpendicular to the wall. Upstream of the fronts OA and AK, $u = 0$. Let w denote the component of the velocity vector perpendicular to the wall. The variation of the momentum of the mass of fluid under consideration in time dt is equal up to quantities small compared to M, to the expression

$$\left[u_1\alpha_1(\vartheta - \chi) + \int\limits_{DB} w\,d\vartheta\right] a_0^2 \rho_0 t\, dt$$

where χ is the angle θ, corresponding to the "triple" point A. Indeed, the difference between the momentum in the regions $O'A'B'D'$ and $OABD$ will be less than $w_{max}\theta\Delta a_0^2\rho_0 tdt$, while the momentum in the region $BB''D''D$ will be less than $w_{max}\theta u a_0^2\rho_0 tdt$. If we denote the angle between the shock wave front and the wall by γ, and dl an element of the curve, then the force impulse of the pressure may be written in the form

$$dt\int\limits_{BDOA}(p-p_1)\cos\gamma\, dl=\Big[\int\limits_{DO}(u-u_1)\,d\Delta-u_1 np_x OA-(u-u_1)\,np_x BD\Big]P_0\, nt\,dt$$

Equating the force impulse to the variation of the momentum, we have

$$u_1\alpha(\vartheta-\chi)+\int\limits_{DB}w d\vartheta=\int\limits_{DO}(u-u_1)\,d\Delta-u_1 np_x OA-(u-u_1)\,np_x BD \quad (3.15)$$

Introducing the notations

$$\chi^\circ=\frac{\chi}{\sqrt{{}^1/_2(n+1)M_0}},\quad \alpha^\circ=\frac{\alpha}{\sqrt{{}^1/_2(n+1)M_0}},\quad W=\frac{w}{a_0M_0\sqrt{{}^1/_2(n+1)M_0}} \quad (3.16)$$

we obtain

$$\mu_1\alpha^\circ(Y-\chi^\circ)+\int\limits_{DB}WdY=\int\limits_{DO}(\mu-\mu_1)\,d\delta-\mu_1(X_A-X_O)-(\mu-\mu_1)(X_B-X_D) \quad (3.17)$$

4. Approximate solution of the problem of a regular reflection of a plane wave from a rough wall at near critical angles of incidence. Let a weak infinitely long plane shock wave, with excess pressure p_1 and with a front that is perpendicular to the wall RE (Fig. 2), be incident on the wall RO the normal of which makes a small angle a with the shock front.

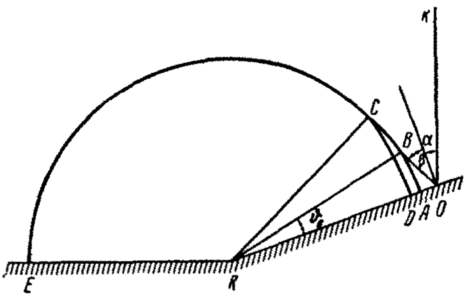

Fig. 2.

The front of the reflected wave $OBCE$ consists, in general, of the interval of the line OB along which the excess pressure is constant, the small arc BC along which a rapid decrease of pressure takes place, and the boundary arc CE which is the front of the wave and along which the pressure hardly differs from the pressure of the incident wave.

In the region $ABCD$ there will be a sharp variation of pressure in the radial direction and also in the direction θ. In this region we can take the flow to correspond to the short wave.

2. Let β be the angle of inclination of the front of the reflected wave to the perpendicular of the wall RO. The pressure in this region (or the number M_0) as well as the angle β are determined from the condition of the equality of the velocity of propagation of incident and reflected waves along the wall:

$$\frac{N_0}{\cos\beta} = \frac{N_1}{\cos\alpha} \tag{4.1}$$

and from the equation of the conservation of momentum perpendicular to the wall. Using (3.17) we obtain

$$M_0 = M_1(1 + \alpha/\beta) \tag{4.2}$$

From equation (4.1), considering the equality (3.1), we have

$$\alpha^2 - \beta^2 = \tfrac{1}{2}(n+1)M_0$$

Hence

$$\alpha = \sqrt{\frac{n+1}{2}M_0}\frac{1-\mu_1}{\sqrt{1-2\mu_1}}, \qquad \beta = \sqrt{\frac{n+1}{2}M_0}\frac{\mu_1}{\sqrt{1-2\mu_1}} \tag{4.3}$$

As μ_1 decreases the angle α decreases and reaches a minimum value for $\mu_1 = 1/3$. This value of α is the critical angle of incidence.

Thus, denoting critical values with asterisks, we have

$$\alpha_* = 2\sqrt{\frac{n+1}{2}M_1} = \frac{2}{\sqrt{3}}\sqrt{\frac{n+1}{2}M_0}, \qquad \beta_* = \frac{1}{2}\alpha_* \tag{4.4}$$

Consequently, a maximum increase of the excess pressure behind the front of the reflected wave is reached for the regular reflection at the critical angle, and it is equal to three.

To every value of α in equations (4.3) correspond two different values of M_0/M_1. In reality, as is well known, a flow establishes itself which corresponds to the value

$$\frac{1}{\mu_1} = 1 + \frac{\alpha^{\vee 2}}{2}\left(1 - \sqrt{1 - \frac{4}{\alpha^{\vee 2}}}\right) \qquad \left(\alpha^{\vee} = \frac{\alpha}{\sqrt{{}^1/_2(n+1)}}\right) \tag{4.5}$$

Formula (4.5) is valid, of course, only for small angles α close to the critical.

The point B in Fig. 2 is the point of intersection of the front of the reflected wave with the sound wave. The equation of the sound wave front and the equation of the reflected wave will be correspondingly

$$\Delta = \frac{n+1}{2}M_0, \quad \Delta + \vartheta\beta - \tfrac{1}{2}\vartheta^2 = \frac{n+1}{4}M_1 + \tfrac{1}{2}\alpha^2 \tag{4.6}$$

Hence

$$\vartheta_B = \sqrt{\frac{n+1}{2}M_0}\left(\frac{\mu_1}{\sqrt{1-2\mu_1}} - \sqrt{1-\mu_1}\right) \tag{4.7}$$

The angle of incidence α_s, at which a sound wave catches up with the front of the incident wave, corresponds to $\theta_B = 0$; beginning with this value of α the region of constant pressure vanishes. From (4.7) we find the corresponding values of μ_1. We have

$$\frac{1}{\mu_1} = \frac{3+\sqrt{5}}{2} \approx 2.65, \qquad \alpha_S = \sqrt{\frac{n+1}{2}M_1}\frac{\sqrt{5}+1}{\sqrt{2(\sqrt{5}-1)}} \approx 2.06\sqrt{\frac{n+1}{2}M_1} \tag{4.8}$$

3. Let us consider the case $\alpha > \alpha_s$. To determine the flow in region $ABCD$ we will make use of the particular solution (2.8). At the front of the sound wave BA $w = u\theta + \vartheta = 0$, consequently $a = 1$.

Along the wall RO for $Y = 0$ we have

$$\frac{dX}{d\mu} = 1 + e^{2c(1-\mu)} + k\sinh 2c(1-\mu)$$

$$\frac{d^2X}{d\mu^2} = -2c\left[e^{2c(1-\mu)} + k\cosh 2c(1-\mu)\right]$$

and $dX/d\mu > 0$ and $d^2X/d\mu^2 < 0$. These conditions are fulfilled for $c > 0$ and $k > -1$.

Constants c and k are chosen in such a manner that for $\mu \to \mu_1$ the front of the reflected wave will become the sonic boundary line $\delta = \mu_1$, so as to satisfy the equation of conservation of momentum (3.17) in the best manner.

The equation of the conservation of momentum perpendicular to the wall in the region which is bounded by the front of the reflected wave, by the line μ = const and by the wall RO, in the case under consideration, gives

$$\begin{aligned}\mu_1\alpha^\circ Y - \frac{1}{c}\tanh c(1-\mu)\frac{Y^2}{2} = (\mu-\mu_1)(X_A - X_B) + \\ + (1-\mu_1)(X_0 - X_D) + \tfrac{1}{2}(1-\mu)(1+\mu-2\mu_1) - \\ - \frac{1}{2c}\{(1+k)(1-\mu_1) - (\mu-\mu_1)[e^{2c(1-\mu)} + k\cosh 2c(1-\mu)\} - \\ - \frac{1}{4c^2}[1 - e^{2c(1-\mu)} - k\sinh 2c(1-\mu)]\end{aligned}$$

where Y is the ordinate of the point of the reflected wave front. This equation is satisfied by all the points of the reflected shock front up to the line $\mu = 0.6$ with an error not greater than 5 per cent, if it is compared with the quantity $\mu_1\alpha^0 Y$. Near $\mu = \mu_1$ the error reaches 20 per cent.

Here $\mu_1 = 0.4$, $a^0 = 1.325$, $c = 2$ and $k = -1$.

In Fig. 3 is shown the velocity field computed in this manner, or more precisely, the constant μ lines, which coincide with the lines of equal pressures, and the front of the reflected wave, are presented.

When the line $\mu = \mu_1$ is approached the pressure gradients decrease, therefore, the flow in this region is not described by the equations of the short waves.

The velocity field near the line $\mu = \mu_1$ is subjected to considerable influence by the flow in the whole region of disturbed motion, which is determined by the acoustic wave equations. For $\mu < 0.5 \div 0.6$ the obtained solution molst likely does not furnish a sufficiently correct picture of the flow.

In the region near the line $\mu = 1$ the velocity field is near $\delta = \mu$, i.e. the pressure along the wall near the point A decreases linearly with distance.

4. When $a_* < a < a_s$, the front of the reflected wave does not have any straight line interval. The velocity field behind the front of the reflected wave can be determined approximately by the use of the particular solution (2.7) of the equations (2.2). The calculations show that the equation of conservation of momentum is best satisfied when $a \to \infty$. In this case the solution (2.7) may be represented in the form

$$X = \mu - \frac{1}{2c} - he^{-2c(\mu-\mu_1)}, \qquad \frac{w}{a_0} = M\vartheta + V = -\frac{\vartheta}{c} \tag{4.9}$$

where h and c are constants. Let M_0 be the Mach number in the point O (Fig. 2) behind the front of the reflected wave. At this point the velocities of propagation of the incident and reflected waves must be equal, therefore

$$\frac{1}{c} = 2 - \mu_1 - \alpha_0^2 - 2he^{-2c(1-\mu_1)} \tag{4.10}$$

The constant h is chosen by virtue of the condition that for $\mu \to \mu_1$ the front of the reflected wave coincides with the sound boundary $\delta = \mu_1$.

Figure 4 gives a picture of the reflection for the case when the angle of incidence is equal to the critical angle. Also $\mu_1 = 1/3$, $a^0 = 2/\sqrt{3}$, $h = 0.65$, $c = 3.2$.

The error which results when equation (3.17) is satisfied, does not exceed 20 per cent even for the values μ near μ_1.

5. Irregular reflection. When the angle $a < a_*$ an irregular reflection takes place, the configuration of which is shown in Fig. 1. In addition to the incident and reflected shock waves there appears a third shock

wave, a Mach wave.

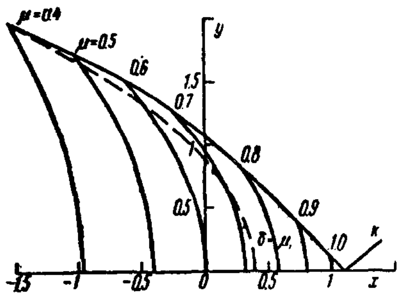

Fig. 3.

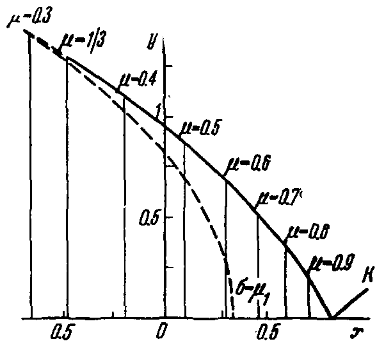

Fig. 4.

Incident and reflected waves now do not intersect at the wall, but in the "triple" point A which moves along the line $\theta = \chi$; the Mach wave connects the point A with the wall.

Let us write the conditions which determine the equation of the velocity of propagation of the shock waves at the point A along the line $\theta = \chi$ and of the propagation of the front of the Mach wave along the wall. Using equations (3.1) we obtain

$$(\alpha^\circ + \chi^\circ)^2 = (\beta^\circ - \chi^\circ) + \mu_2, \quad (\alpha^\circ + \chi^\circ)^2 - \psi_A{}^{\circ 2} = \mu_2 - \mu_1, \quad \psi_A{}^\circ = \frac{\psi_A}{\sqrt{{}^1/_2(n+1)M_0}} \tag{5.1}$$

where M_0 is the number M at the origin of the Mach wave, ψ_A is the angle between the normal to the Mach wave at the point A and the ray $\theta = \chi$.

When comparing the velocity of propagation of the Mach wave along the wall and the velocity of motion of the point A parallel to the wall, we obtain

$$\alpha^{\circ 2} = 1 - \mu_1 + 2(X_A - X_0) \tag{5.2}$$

Hence $\alpha^{\circ 2} = 1 - \mu_1$ for $\chi \to 0$, therefore

$$M_0 / M_1 = 1 + \alpha^{\vee 2} \tag{5.3}$$

When the angle of incidence is equal to the critical angle, $\alpha^\vee = 2$, and, consequently, for this value of the angle the excess pressure at the origin of the Mach wave exceeds five times the excess pressure in the incident wave. If the angle α is near zero, then the equation of the reflected wave will differ little from the equation of the sound periphery:

$$X + \tfrac{1}{2} Y^2 = \mu_1 \tag{5.4}$$

The equation of the front of the incident wave may be presented in the form

$$X-\alpha^\circ Y=\tfrac{1}{2}\mu_1+\tfrac{1}{2}\alpha^{\circ 2} \tag{5.5}$$

Hence, for the angle, which determines the position of the triple point, we obtain the relation

$$\chi=\sqrt{{}^1/_2(n+1)M_1}-\alpha \quad \text{or} \quad \chi^\vee=1-\alpha^\vee \tag{5.6}$$

Thus, if the reflected wave were near the sound wave, the angle χ would become zero for an angle of incidence equal to half the critical angle. Actually, angle χ becomes zero, when the angle of incidence is equal to the critical angle, and it is of very small magnitude roughly in the interval $0.5\, a_* < a < a_*$. In this range a sharp variation of pressure takes place in an extremely small region near the Mach wave.

To describe the picture of flow in this region we may, therefore, make use of the equations (2.9). The position of the origin of the Mach wave is given by the equation $X = 0.5$, i.e. the value δ_0 in equations (2.9) should be taken to be equal to 0.5. We will assume

$$X=\tfrac{1}{2}+\varepsilon(\xi-\xi_0), \qquad Y=\varepsilon\eta \tag{5.7}$$

where ϵ is a small quantity, and ξ_0 is a constant equal to the value ξ for $\mu = 1$. With the accuracy up to the order of magnitude ϵ^2 we have $\delta = X$. Using the above notation the solution (2.11) assumes the form:

$$\xi-\xi_0=2\mu-1+\tfrac{1}{2}\eta^2, \qquad \nu=-(\mu-\tfrac{1}{2})\eta-\tfrac{1}{6}\eta^3 \tag{5.8}$$

When solving the last of these equations for η, we have

$$\eta=\eta_*(\nu,\ \mu) \tag{5.9}$$

Upon adding the particular solutions (2.10) and (2.11) and assuming $c^2 = \epsilon h$, we obtain

$$\eta=\eta_*(\mu,\nu)-h(a-\mu)\nu, \ \ \xi-\xi_0=2\mu-1+h\left[(\tfrac{1}{2}-a)(a-\mu)^2+\tfrac{2}{3}(a-\mu)^3\right] \tag{5.10}$$

Computing the curvature of the lines μ = const for $\eta = 0$ for the velocity field which is given by the equations (5.10), we obtain

$$\left(\frac{d^2\xi}{d\eta^2}\right)_{\eta=0}=\frac{h+4(2\mu-1)^{-2}}{h^2[a-\mu+2/h(2\mu-1)]^2} \tag{5.11}$$

Since the equation of conservation of momentum (3.17) is to be satisfied along the Mach wave in the best possible manner, it follows that the curvature of the curve $\mu = 1$ for $\eta = 0$ should be taken as inifinite. Also $a = 1 - 2/h$.

From formula (5.11) we then obtain

$$\left(\frac{d^2\xi}{d\eta^2}\right)_{\eta=0}=\frac{1}{(1-\mu)^2}\,\frac{4+(2\mu-1)^2h}{[4+(2\mu-1)h]^2} \tag{5.12}$$

Solution (5.10) will now be of the form:

$$\eta = [2 - (1-\mu) h] \nu + \eta_*(\mu, \nu)$$
$$\xi - \xi_0 = -(1-\mu)^2\left[1 + \frac{h}{3}(\mu - \tfrac{1}{4})\right] + \tfrac{1}{2}[h\nu^2 + \eta_*^2(\mu, \nu)] \qquad (5.13)$$

Also (5.14)

$$(\partial\xi/\partial\mu)_{\nu=0} = (1-\mu)[4 + h(2\mu - 1)], \qquad (\partial^2\xi/\partial\mu^2)_{\nu=0} = -4[1 + h(\mu - \tfrac{3}{4})]$$

If along the wall the quantity $\partial\xi/\partial\mu$ is to increase when μ decreases the inequality $h < 4$ must be satisfied. The variation of the quantity h influences only insignificantly the distribution of the pressure along the wall, but it changes noticeably the curvature of the lines μ = const for values of μ near unity. If further the absolute magnitude of ν is to increase along the lines μ = const, as η increases the value of h should be greater than two. Figure 5 shows the velocity field for $h = 2\ 2/3$, for which the equation of conservation of momentum in the region adjacent to the Mach wave is satisfied with maximum accuracy. The equations which determine the position of the Mach wave and the equations of the reflected wave in the case under consideration have the following form, respectively:

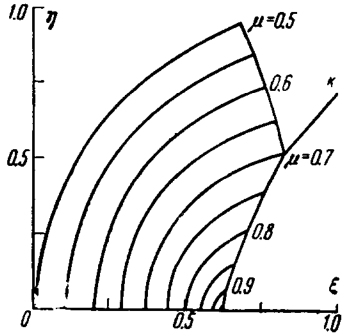

Fig. 5.

$$\frac{d\xi}{d\eta} = \sqrt{1-\mu}, \qquad \frac{d\xi}{d\eta} = \sqrt{1-\mu-\mu_1}$$

When the value of the angle α is very near, but less than, the critical value, the velocity field behind the front of the reflected wave will be very near the velocity field for the angles of incidence, which are just a little greater than the critical angle, with the exception of a very small region near the point of intersection of the front of the incident wave and the wall. The size of this region approaches zero for $\alpha \to \alpha_*$. In this small region which includes the Mach wave the velocity field will coincide approximately with the velocity field represented in Fig. 5. In the construction of the velocity field behind the reflected wave we will again make use of the solution (2.7). We assume that $a = m$, where m is equal to the value μ for which the variation of the pressure near the point O begins to be sharp. Because the size of the region of the sharp pressure variation may be considered to be insignificantly small, we can equate the velocities of propagation of the front of the reflected and incident waves along the wall and obtain:

$$c = \frac{2(1+k)}{2m-1}$$

The magnitude k is chosen in such a way that the front of the reflected wave for $\mu \to \mu_1$ will tend toward the sound periphery $\delta = \mu_1$. Along the front of the reflected wave the equation of conservation of momentum (3.17) must be satisfied. Calculations show that in order to satisfy this condition for $\mu_1 = 0.25$ the quantity c must be of the order of 5. Figure 6 shows the construction of the velocity field for $c = 5.25$, $k = 0.05$, corresponding to the value $m = 0.7$. In this case the equation of conservation of momentum is satisfied with an accuracy of 10 per cent relative to the term $\mu_1 a^0 Y$. For $\mu_1 = 0.3$ the error increases when using the same values of c and k. For $\mu_1 = 0.2$, the number m may be taken to be equal to 0.65. The velocity field, given in Fig. 6, by approximation may be considered to correspond to the velocity field behind the front of the reflected wave in the interval $0.2 \leqslant \mu \leqslant 0.3$ which corresponds to the interval of angle a variation from $a_* = 2\sqrt{1/2(n+1)M_1}$ to $a = 1.41\sqrt{1/2(n+1)M_1}$.

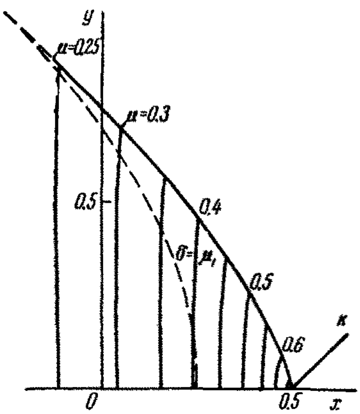

Fig. 6.

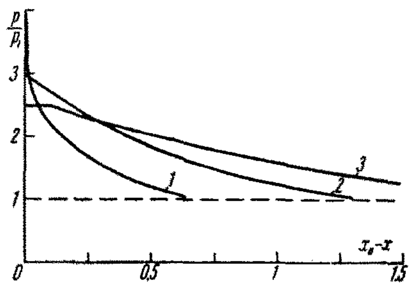

Fig. 7.

The approximate solution given above does not permit us to determine the dimensions of the region of the strong pressure variation, i.e. the value ϵ nor the exact boundary which separates the region of the very sharp pressure variations near the Mach wave. For this, it is necessary to derive the solution of the system of equations (1.19), which will include the special feature that describes the pressure field near the Mach wave.

If the dependence $\chi = \chi\,(a)$ is determined on the basis of optical measurements, then the magnitude ϵ is determined from the equations

$$\chi = 0.5\,\varepsilon\sqrt{\mu_1}\sqrt{\frac{n+1}{2}M_1}, \qquad \frac{1}{\mu} = 1 + \frac{\alpha^2}{{}^1/_2\,(n+1)\,M_1}$$

Figure 7 shows the curves 1, 2, and 3 of the pressure distribution for the angles of incidence $a = 1.73\sqrt{1/2(n+1)M_1}$, the critical angle $a_* = 2\sqrt{1/2(n+1)M_1}$ and for the angle $a = 2.095\sqrt{1/2(n+1)M_1}$. The plotted lines show that the regions of increased pressure near the critical angle are of very small extent and therefore, for their experimental observation in experiments with explosions of small charges, instruments of very high resolving power are necessary.

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Translated by J.R.W.